BV QUANTIZATION OF A GENERIC DEGENERATE QUADRATIC LAGRANGIAN

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arXiv:hep-th/0403201v1 20 Mar 2004

Generalizing the Yang–Mills gauge theory, we provide the BV quantization of a field model with a generic almost-regular quadratic Lagrangian by use of the fact that the configuration space of such a field model is split into the gauge-invariant and gauge-fixing parts.

The Batalin–Vilkoviski (henceforth BV) quantization [2, 6] provides the universal scheme of quantization of gauge-invariant Lagrangian field theories. Given a classical Lagrangian, BV quantization enables one to obtain a gauged-fixed BRST invariant Lagrangian in the generating functional of perturbation quantum field theory. However, the BV quantization scheme does not automatically provide the path-integral measure, unless a gauge model is irreducible. We apply this scheme to a generic degenerate (almost-regular) quadratic Lagrangian.

We follow the geometric formulation of classical field theory where classical fields are represented by sections of fiber bundles. Let $Y \to X$ be a smooth fiber bundle provided with bundle coordinates (x^μ, y^i). The configuration space of a first order Lagrangian field theory on Y is the first order jet manifold J^1Y of Y equipped with the adapted coordinates (x^μ, y^i, y^i_μ), where y^i_μ are coordinates of derivatives of fields [4]. A first-order Lagrangian is defined as a density

$$L = \mathcal{L} dx : J^1Y \to \overset{n}{\wedge} T^*X, \qquad n = \dim X, \tag{1}$$

on J^1Y. The corresponding Euler–Lagrange equations are given by the subset

$$\delta_i \mathcal{L} = (\partial_i - d_\lambda \partial^\lambda_i)\mathcal{L} = 0, \qquad d_\lambda = \partial_\lambda + y^i_\lambda \partial_i + y^i_{\lambda\mu}\partial^\mu_i, \tag{2}$$

of the second-order jet manifold J^2Y of Y coordinated by $(x^\mu, y^i, y^i_\lambda, y^i_{\lambda\mu})$. Any Lagrangian L (1) yields the Legendre map

$$\widehat{L} : J^1Y \underset{Y}{\longrightarrow} \Pi, \qquad p^\lambda_i \circ \widehat{L} = \partial^\lambda_i \mathcal{L}, \tag{3}$$

of the configuration space J^1Y to the momentum phase space

$$\Pi = \overset{n}{\wedge} T^*X \underset{Y}{\otimes} V^*Y \underset{Y}{\otimes} TX \to Y, \tag{4}$$

called the Legendre bundle and equipped with the holonomic bundle coordinates $(x^\lambda, y^i, p^\mu_i)$. By VY and V^*Y are denoted the vertical tangent and cotangent bundles of $Y \to X$, respectively.

Let us consider a quadratic Lagrangian

$$\mathcal{L} = \frac{1}{2} a^{\lambda\mu}_{ij} y^i_\lambda y^j_\mu + b^\lambda_i y^i_\lambda + c, \tag{5}$$

where a, b and c are local functions on Y. This property is coordinate-independent since $J^1Y \to Y$ is an affine bundle modelled over the vector bundle $T^*X \underset{Y}{\otimes} VY$. The key point is that, if a Lagrangian L (5) is almost-regular, it can be brought into the Yang–Mills type form as follows.

Given $\mathcal{L}$ (5), the associated Legendre map (3) reads

$$p^\lambda_i \circ \widehat{L} = a^{\lambda\mu}_{ij} y^j_\mu + b^\lambda_i. \tag{6}$$

Let a Lagrangian $\mathcal{L}$ (5) be almost-regular, i.e., the matrix function a is a linear bundle morphism

$$a : T^*X \underset{Y}{\otimes} VY \to \Pi, \qquad p^\lambda_i = a^{\lambda\mu}_{ij} \overline{y}^j_\mu, \tag{7}$$

of constant rank, where $(x^\lambda, y^i, \overline{y}^i_\lambda)$ are holonomic bundle coordinates on $T^*X \underset{Y}{\otimes} VY$. Then the image N_L of $\widehat{L}$ (6) is an affine subbundle of the Legendre bundle (4). Hence, $N_L \to Y$ has a global section. For the sake of simplicity, let us assume that it is the canonical zero section $\widehat{0}(Y)$ of $\Pi \to Y$. The kernel of the Legendre map (6) is also an affine subbundle of the affine jet bundle $J^1Y \to Y$. Therefore, it admits a global section

$$\Gamma : Y \to \mathrm{Ker}\,\widehat{L} \subset J^1Y, \qquad a^{\lambda\mu}_{ij}\Gamma^j_\mu + b^\lambda_i = 0, \tag{8}$$

which is a connection on $Y \to X$. With such a connection, the Lagrangian (5) is brought into the form

$$\mathcal{L} = \frac{1}{2} a^{\lambda\mu}_{ij} (y^i_\lambda - \Gamma^i_\lambda)(y^j_\mu - \Gamma^j_\mu) + c'. \tag{9}$$

Let us refer to the following theorems [4, 5].

Theorem 1: There exists a linear bundle morphism

$$\sigma : \Pi \underset{Y}{\to} T^*X \underset{Y}{\otimes} VY, \qquad \overline{y}^i_\lambda \circ \sigma = \sigma^{ij}_{\lambda\mu} p^\mu_j, \tag{10}$$

such that

$$a \circ \sigma \circ a = a, \qquad a^{\lambda\mu}_{ij}\sigma^{jk}_{\mu\alpha}a^{\alpha\nu}_{kb} = a^{\lambda\nu}_{ib}. \tag{11}$$

The equalities (8) and (11) give the relation $(a\circ\sigma_0)^{\lambda j}_{i\mu}b^\mu_j = b^\lambda_i$. Note that the morphism σ (10) is not unique, but it falls into the sum $\sigma = \sigma_0 + \sigma_1$ such that

$$\sigma_0 \circ a \circ \sigma_0 = \sigma_0, \qquad a\circ\sigma_1 = \sigma_1 \circ a = 0, \tag{12}$$

where σ_0 is uniquely defined.

Theorem 2: There is the splitting

$$J^1Y = \mathrm{Ker}\,\widehat L \underset{Y}{\oplus} \mathrm{Im}(\sigma_0\circ \widehat L), \tag{13}$$

$$y^i_\lambda = \mathcal{S}^i_\lambda + \mathcal{F}^i_\lambda = [y^i_\lambda - \sigma_0{}^{ik}_{\lambda\alpha}(a^{\alpha\mu}_{kj}y^j_\mu + b^\alpha_k)] + [\sigma_0{}^{ik}_{\lambda\alpha}(a^{\alpha\mu}_{kj}y^j_\mu + b^\alpha_k)]. \tag{14}$$

The relations (12) lead to the equalities

$$\sigma_0{}^{jk}_{\mu\alpha}\mathcal{R}^\alpha_k = 0, \qquad \mathcal{F}^i_\mu = (\sigma_0\circ a)^{i\lambda}_{\mu j}(y^j_\lambda - \Gamma^j_\lambda). \tag{15}$$

By virtue of these equalities the Lagrangian (5) takes the Yang–Mills type form

$$\mathcal{L} = \frac{1}{2}a^{\lambda\mu}_{ij}\mathcal{F}^i_\lambda\mathcal{F}^j_\mu + c'. \tag{16}$$

Indeed, let us consider gauge theory of principal connections on a principal bundle $P\to X$ with a structure Lie group G. Principal connections on $P\to X$ are represented by sections of the affine bundle $C = J^1P/G \to X$, modelled over the vector bundle $T^*X\otimes V_GP$ [4]. Here, $V_GP = VP/G$ is the fiber bundle in Lie algebras $\mathfrak{g}$ of the group G. Given the basis $\{\varepsilon_r\}$ for $\mathfrak{g}$, we obtain the local fiber bases $\{e_r\}$ for V_GP. The connection bundle C is coordinated by (x^μ, a^r_μ) such that, written relative to these coordinates, sections $A = A^r_\mu dx^\mu \otimes e_r$ of $C\to X$ are the familiar local connection one-forms, regarded as gauge potentials. The configuration space of gauge theory is the jet manifold J^1C equipped with the coordinates $(x^\lambda, a^m_\lambda, a^m_{\mu\lambda})$. It admits the canonical splitting

$$a^r_{\mu\lambda} = \mathcal{S}^r_{\mu\lambda} + \mathcal{F}^r_{\mu\lambda} = \frac{1}{2}(a^r_{\mu\lambda} + a^r_{\lambda\mu} - c^r_{pq}a^p_\mu a^q_\lambda) + \frac{1}{2}(a^r_{\mu\lambda} - a^r_{\lambda\mu} + c^r_{pq}a^p_\mu a^q_\lambda) \tag{17}$$

(cf. (14)), where $\mathcal{F}$ is the strength of gauge fields up to the factor 1/2. The Yang–Mills Lagrangian on the configuration space J^1C reads

$$L_{\mathrm{YM}} = a^G_{pq}g^{\lambda\mu}g^{\beta\nu}\mathcal{F}^p_{\lambda\beta}\mathcal{F}^q_{\mu\nu}\sqrt{|\,g\,|}\,dx, \qquad g = \det(g_{\mu\nu}), \tag{18}$$

where a^G is a non-degenerate G-invariant metric in the dual of the Lie algebra of $\mathfrak{g}$ and g is a non-degenerate metric on X.

If the Lagrangian (16) possesses no gauge symmetries, its quantization in the framework of perturbation quantum field theory can be given by the generating functional

$$Z = N^{-1}\int \exp\{\int(\mathcal{L} + iJ_i y^i)dx\}\prod_x [dy(x)]$$

of Euclidean Green functions.

Let us suppose that the Lagrangian L (16) is invariant under some gauge group G_X of vertical automorphisms of the fiber bundle $Y \to X$ which acts freely on the space of sections of $Y \to X$. Its infinitesimal generators are represented by vertical vector fields $u = u^i(x^\mu, y^j)\partial_i$ on $Y \to X$ which give rise to the vector fields

$$J^1u = u^i\partial_i + d_\lambda u^i\partial_i^\lambda, \qquad d_\lambda = \partial_\lambda + y^i_\lambda\partial_i, \tag{19}$$

on J^1Y. Let us also assume that G_X is indexed by m parameter functions $\xi^r(x)$ such that $u = u^i(x^\lambda, y^j, \xi^r)\partial_i$, where

$$u^i(x^\lambda, y^j, \xi^r) = u^i_r(x^\lambda, y^j)\xi^r + u^{i\mu}_r(x^\lambda, y^j)\partial_\mu\xi^r \tag{20}$$

are linear first order differential operators on the space of parameters $\xi^r(x)$. The vector fields $u(\xi^r)$ must satisfy the commutation relations

$$[u(\xi^q), u(\xi'^p)] = u(c^r_{pq}\xi'^p\xi^q),$$

where c^r_{pq} are structure constants. The Lagrangian L (16) is gauge-invariant iff its Lie derivative $\mathbf{L}_{J^1u}L$ along the vector fields (19) vanishes, i.e.,

$$(u^i\partial_i + d_\lambda u^i\partial_i^\lambda)\mathcal{L} = 0. \tag{21}$$

In order to study the invariance condition (21), let us consider the Lagrangian (5) written in the form (9). Since

$$J^1u(y^i_\lambda - \Gamma^i_\lambda) = \partial_k u^i(y^k_\lambda - \Gamma^k_\lambda), \tag{22}$$

one easily obtains from the equality (21) that

$$u^k\partial_k a^{\lambda\mu}_{ij} + \partial_i u^k a^{\lambda\mu}_{kj} + a^{\lambda\mu}_{ik}\partial_j u^k = 0. \tag{23}$$

It follows that the summands of the Lagrangian L (9) and, consequently, the summands of the Lagrangian (16) are separately gauge-invariant, i.e.,

$$J^1u(a^{\lambda\mu}_{ij}\mathcal{F}^i_\lambda\mathcal{F}^j_\mu) = 0, \qquad J^1u(c') = u^k\partial_k c' = 0. \tag{24}$$

The equalities (15), (22) and (23) give the transformation law

$$J^1u(a^{\lambda\mu}_{ij}\mathcal{F}^j_\mu) = -\partial_i u^k a^{\lambda\mu}_{kj}\mathcal{F}^j_\mu. \tag{25}$$

The relations (12) and (23) lead to the equality

$$a^{\lambda\mu}_{ij}[u^k\partial_k\sigma_0{}^{jn}_{\mu\alpha} - \partial_k u^j\sigma_0{}^{kn}_{\mu\alpha} - \sigma_0{}^{jk}_{\mu\alpha}\partial_k u^n]a^{\alpha\nu}_{nb} = 0. \tag{26}$$

For the sake of simplicity, let us assume that the gauge group G_X preserves the splitting (13), i.e., its infinitesimal generators u obey the condition

$$u^k\partial_k(\sigma_0{}^{im}_{\lambda\nu}a^{\nu\mu}_{mj}) + \sigma_0{}^{im}_{\lambda\nu}a^{\nu\mu}_{mk}\partial_j u^k - \partial_k u^i\sigma_0{}^{km}_{\lambda\nu}a^{\nu\mu}_{mj} = 0. \tag{27}$$

The relations (22) and (27) lead to the transformation law

$$J^1u(\mathcal{F}^i_\mu) = \partial_j u^i\mathcal{F}^j_\mu. \tag{28}$$

Since $\mathcal{S}^i_\lambda = y^i_\lambda - \mathcal{F}^i_\lambda$, one can easily derive from the formula (28) the transformation law

$$J^1u(\mathcal{S}^i_\mu) = d_\lambda u^i - \partial_j u^i\mathcal{F}^j_\lambda = d_\lambda u^i - \partial_j u^i(y^j_\lambda - \mathcal{S}^j_\lambda) = \partial_\lambda u^i + \partial_j u^i\mathcal{S}^j_\lambda \tag{29}$$

of $\mathcal{S}$. A glance at this expression shows that the gauge group G_X acts freely on the space of sections $\mathcal{S}(x)$ of the fiber bundle $\operatorname{Ker}\widehat{L} \to Y$ in the splitting (14). Then some combinations $b^r{}^\mu_i\mathcal{S}^i_\mu$ of $\mathcal{S}^i_\mu$ can be used as the gauge-fixing condition

$$b^r{}^\mu_i\mathcal{S}^i_\mu(x) = \alpha^r(x), \tag{30}$$

similar to the generalized Lorentz gauge in Yang–Mills gauge theory.

Turn now to the BV quantization of a Lagrangian system with the gauge-invariant Lagrangian $\mathcal{L}$ (16). We follow the quantization procedure in [2, 6] reformulated in the jet terms [1, 3]. Note that odd fields C^r can be introduced as the basis for a graded manifold determined by the dual E^* of a vector space $E \to X$ coordinated by (x^λ, e^r). Then the k-order jets $C^r_{\lambda_k\ldots\lambda_1}$ are defined as the basis for a graded manifold determined by the dual of the k-order jet bundle $J^kE \to X$, which is a vector bundle [7, 8]. The BV quantization procedure falls into the two steps. At first, one obtains a proper solution of the classical master equation and, afterwards, the gauge-fixed BRST invariant Lagrangian is constructed.

Let the number m of parameters of the gauge group G_X do not exceed the fiber dimension of $\operatorname{Ker}\widehat{L} \to Y$. Then we can follow the standard BV procedure for irreducible gauge theories in [6].

Firstly, one should introduce odd ghosts C^r of ghost number 1 together with odd antifields y^*_i of ghost number -1 and even antifields C^*_r of ghost number -2. Then a proper solution of the classical master equation reads

$$\mathcal{L}_{\rm PS} = \mathcal{L} + y^*_i u^i_C - \frac{1}{2} c^r_{pq} C^*_r C^p C^q, \tag{31}$$

where u_C is the vector field

$$u_C = u^i_r(x^\lambda, y^j) C^r + u^{i\mu}_r(x^\lambda, y^j) C^r_\mu \tag{32}$$

obtained from the vector field (20) by replacement of parameter functions ξ^r and its derivatives $\partial_\mu \xi^r$ with the ghosts C^r and their jets C^r_μ.

Secondly, one introduces the gauge-fixing density depending on fields y^i, ghosts C^r and additional auxiliary fields, which are odd fields $\overline{C}_r$ of ghost number -1 and even fields B_r of zero ghost number. Passing to the Euclidean space-time, this gauge-fixing density reads

$$\Psi = \overline{C}_p(\frac{i}{2} h^{pr} B_r + b^{p\mu}_i \mathcal{S}^i_\mu), \tag{33}$$

where $h^{pr}(x)$ is a non-degenerate positive-definite matrix function on X and $b^{p\mu}_i \mathcal{S}^i_\mu$ are gauge-fixing combinations (30).

Thirdly, the desired gauge-fixing Lagrangian $\mathcal{L}_{\rm GF}$ is derived from the extended Lagrangian

$$\mathcal{L}'_{\rm PS} = \mathcal{L}_{\rm PS} + i\overline{C}^{*p} B_p,$$

where $\overline{C}^{*p}$ are antifields of auxiliary fields $\overline{C}_p$, by replacement of antifields with the variational derivatives

$$y^*_i = \frac{\delta\Psi}{\delta y^i}, \qquad C^*_p = \frac{\delta\Psi}{\delta C^p} = 0, \qquad \overline{C}^{*p} = \frac{\delta\Psi}{\delta\overline{C}_p} = \frac{i}{2} h^{pr} B_r - b^{p\mu}_i \mathcal{S}^i_\mu \tag{34}$$

(see the formula (2)). We obtain

$$\mathcal{L}_{\rm GF} = \mathcal{L} + \delta_i \Psi u^i_C - B_p(\frac{1}{2} h^{pr} B_r - i b^{p\mu}_i \mathcal{S}^i_\mu). \tag{35}$$

Let us bring its second term into the form

$$(\partial_i\Psi - d_\lambda \partial^\lambda_i \Psi) u^i_C = \partial_i \Psi u^i_C + \partial^\lambda_i \Psi d_\lambda(u^i_C) - d_\lambda(\partial^\lambda_i \Psi u^i_C) =$$
$$J^1 u_C(\Psi) - d_\lambda(\partial^\lambda_i \Psi u^i_C),$$

where

$$J^1 u_C = u^i_C \partial_i + d_\lambda u^i_C \partial^\lambda_i, \qquad d_\lambda = \partial_\lambda + y^i_\lambda \partial_i + C^r_\lambda \frac{\partial}{\partial C^r} \tag{36}$$

is the jet prolongation of the vector field u_C (32). In view of the transformation law (29), we have

$$J^1u_C(\Psi) = -\overline{C}_p b^{p\lambda}_i J^1u_C(\mathcal{S}^i_\lambda) = -\overline{C}_p b^{p\lambda}_i [\partial_\lambda u^i_r C^r + u^i_r C^r_\lambda + \partial_\lambda u^{i\mu}_r C^r_\mu + u^{i\mu}_r C^r_{\lambda\mu} + (\partial_j u^i_r C^r + \partial_j u^{i\mu}_r C^r_\mu)\mathcal{S}^j_\lambda] = -\overline{C}_p M^p_r C^r, \tag{37}$$

where $M^p_r C^r$ is a second order differential operator on ghosts C^r. Then the gauge-fixing Lagrangian (35) up to a divergence term takes the form

$$\mathcal{L}_{\rm GF} = \mathcal{L} - \overline{C}_p M^p_r C^r - \frac{1}{2}h^{pr}B_pB_r + iB_p b^{p\mu}_i \mathcal{S}^i_\mu. \tag{38}$$

Finally, one can write the generating functional

$$Z = N^{-1}\int \exp\{\int(\mathcal{L} - \overline{C}_p M^p_r C^r - \frac{1}{2}h^{pr}B_pB_r + iB_p b^{p\mu}_i \mathcal{S}^i_\mu + iJ_k y^k)dx\} \prod_x [dB_p][d\overline{C}][dC][dy]$$

of Euclidean Green functions. Integrating it as a Guassian integral with respect to the variables B_p, we obtain

$$Z = N'^{-1}\int \exp\{\int(\mathcal{L} - \overline{C}_p M^p_r C^r - \frac{1}{2}h^{-1}_{pr} b^{p\mu}_i b^{r\nu}_j \mathcal{S}^i_\mu \mathcal{S}^j_\nu + iJ_k y^k)dx\} \prod_x [d\overline{C}][dC][dy]. \tag{39}$$

Of course, the Lagrangian

$$\mathcal{L} - \overline{C}_p M^p_r C^r - \frac{1}{2}h^{-1}_{pr} b^{p\mu}_i b^{r\nu}_j \mathcal{S}^i_\mu \mathcal{S}^j_\nu \tag{40}$$

in the generating functional (39) is not gauge-invariant, but it is invariant under the BRST transformation

$$\vartheta = u^i_C\partial_i + d_\lambda u^i_C \partial^\lambda_i + \overline{v}_r \frac{\partial}{\partial \overline{C}_r} + v^r\frac{\partial}{\partial C^r} + d_\lambda v^r \frac{\partial}{\partial C^r_\lambda} + d_\mu d_\lambda v^r \frac{\partial}{\partial C^r_{\mu\lambda}}, \tag{41}$$

$$d_\lambda = \partial_\lambda + y^i_\lambda \partial_i + y^i_{\lambda\mu}\partial^\mu_i + C^r_\lambda \frac{\partial}{\partial C^r} + C^r_{\lambda\mu}\frac{\partial}{\partial C^r_\mu},$$

whose components v are given by the antibrackets

$$v^r = (C^r, \mathcal{L}'_{\rm PS}) = \frac{\delta \mathcal{L}'_{\rm PS}}{\delta C^*_r} = -\frac{1}{2}c^r_{pq}C^pC^q, \qquad \overline{v}_r = (C^r, \mathcal{L}'_{\rm PS}) = \frac{\delta \mathcal{L}'_{\rm PS}}{\delta \overline{C}^{*r}} = iB_r$$

restricted to the shell (34) and to the solution $B_r = ih^{-1}_{rp} b^{p\mu}_i \mathcal{S}^i_\mu$ of the Euler–Lagrange equations $\delta\mathcal{L}_{\rm GF}/\delta B_r = 0$.

For instance, the generating functional (39) in the case of $\mathcal{S}^r_{\mu\lambda}$ (17), $h_{pr} = a^G_{pr}$ and $b^{p\nu\mu}_r = \delta^p_r g^{\nu\mu}$ restarts the familiar BV quantization of the Yang–Mills gauge theory.

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